On the Coefficients of the Map $\Psi : \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}} \to \widehat{\mathbb{C}} \setminus M$

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Abstract

We consider the coefficients of the conformal mapping Ψ given by Douady and Hubbard from the exterior of unit disk onto the exterior of the Mandelbrot set. As a consequence, we prove Don Zagier's conjecture. We also extend a dynamical system to higher degrees.

1. Introduction.

In [D-H], Douady and Hubbard proved that the Mandelbrot set M is connected by constructing an analytic homeomorphism

$$\Psi(z) = z + \sum_{m=0}^{\infty} b_m z^{-m} \tag{1}$$

of $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ onto $\widehat{\mathbb{C}} \setminus M$, where $D = \{w \in \mathbb{C} : |w| < 1\}$. Jungreis [J] presented an algorithm which obtains the coefficients b_m of this map. Since then the detailed properties of these coefficients have been investigated by Ewing–Schober [E-S90, E-S92] and Bielefeld–Fisher–Haeseler(–Zagier) [B-F-H].

In this notes, we first implement Jungreis' algorithm on an actual computer in an efficient way. And then we extend Ewing–Schober's results on higher degrees.

2. Jungreis' algorithm.

We recursively define a sequence of polynomials $\{A_n\}_n$ by

$$A_0(c) = c, \quad A_{n+1}(c) = [A_n(c)]^2 + c. \tag{2}$$

Evidently, A_n is a monic polynomial of degree 2^n. Then we can describe M as follows.

$$M = \{c \in \mathbb{C} : \lim_{n\to\infty} A_n(c) \not\to \infty\}.$$

The maps Φ_n defined by $\Phi_n \equiv A_n^{1/2^n}$ converge locally uniformly in $\widehat{\mathbb{C}} \setminus M$ to $\Phi = \Psi^{-1}$, the inverse of the mapping Ψ (see [J]). Since the maps Φ_n are one-to-one near ∞, we can define their inverse maps $\Psi_n \equiv \Phi_n^{-1}$. Evidently, we have

$$A_n(\Psi_n(z)) = z^{2^n}. \tag{3}$$

Our main concern is to calculate the coefficients b_m in the equation (1) explicitly. From (1) and (2), we obtain

$$A_n(\Psi(z)) = z^{2^n} + \sum_{m=0}^{\infty} (2^n b_m + C_m) z^{2^n - m - 1}$$

where $C_m \in \mathbb{Z}[b_0, \ldots, b_{m-1}]$. Thus, all coefficients b_m are diadic rationals.

Lemma 1. ([J] Corollary to Theorem 4)

$$\Psi(z) = \Psi_n(z) + O(1/z^{2^{n+1}-2}), \quad |z| \gg 1. \tag{4}$$

Using (3) and (4), Jungreis proposed the following algorithm.

1. Suppose we want to obtain the values $b_0, \ldots, b_l$. Let
$$\widehat{\Psi}(z) = z + \sum_{m=0}^{l} b_m z^{-m}.$$
2. Choose an integer $k \geq 1$ satisfying $l \leq 2^{k+1} - 3$.
3. Substitute $c = \widehat{\Psi}(z)$ in $A_k(c)$ in (2), and compute the coefficients of $z^{2^k-1}, \ldots, z^{2^k-l-1}$.
4. Then, we can compute the values of $b_0, \ldots, b_l$ recursively.

Example: $l = 1$.

1. Let $l = 1$, so $\widehat{\Psi}(z) := z + b_0 + b_1 z^{-1}$.
2. Choose $k = 1$.
3. Then, $A_1(\widehat{\Psi}(z)) = z^2 + (2b_0 + 1)z + (b_0^2 + b_0 + 2b_1) + (2b_0 b_1 + b_1)z^{-1} + b_1^2 z^{-2}$.
4. From $2b_0 + 1 = 0$, we obtain $b_0 = -1/2$, and from $b_0^2 + b_0 + 2b_1 = 0$, we obtain $b_1 = 1/8$.

It is easy to see that the straight implimentation of Jungreis' algorithm requires huge amount of memory in order store to all previous terms. However, noticing that b_m's satisfy the equation

$$b_m^{(k)} = b_m + \sum_{i=-2^{k-1}}^{m+2^{k+1}} b_i^{(k-1)} b_{m-i}^{(k-1)},$$

where $b_m^{(n)}$ is the coefficient of z^{-m} in $A_n(\Psi(z))$, we can write a very efficient program.

In this way, we compute the coefficients $b_0, \ldots, b_{2045}$ of Ψ, and obtain

$$\Psi(z) = z - \frac{1}{2^1} + \frac{1}{2^3 z} - \frac{1}{2^2 z^2} + \frac{15}{2^7 z^3} - \frac{47}{2^{10} z^5} - \frac{1}{2^4 z^6} + \frac{987}{2^{15} z^7} - \frac{3673}{2^{18} z^9} + \frac{1}{2^5 z^{10}} + \cdots$$

Figure 1 is the image of Ψ on the polar coordinates with $r = 1.0001$, 1.05, 1.1, 1.15, 1.2, 1.25, 1.3, 1.35, 1.4, 1.45, and 1.5, and $\theta = 2\pi i/240 (i = 0, \ldots, 239)$, using the approximation $\Psi(z) = z + \sum_{m=0}^{509} b_m z^{-m}$.

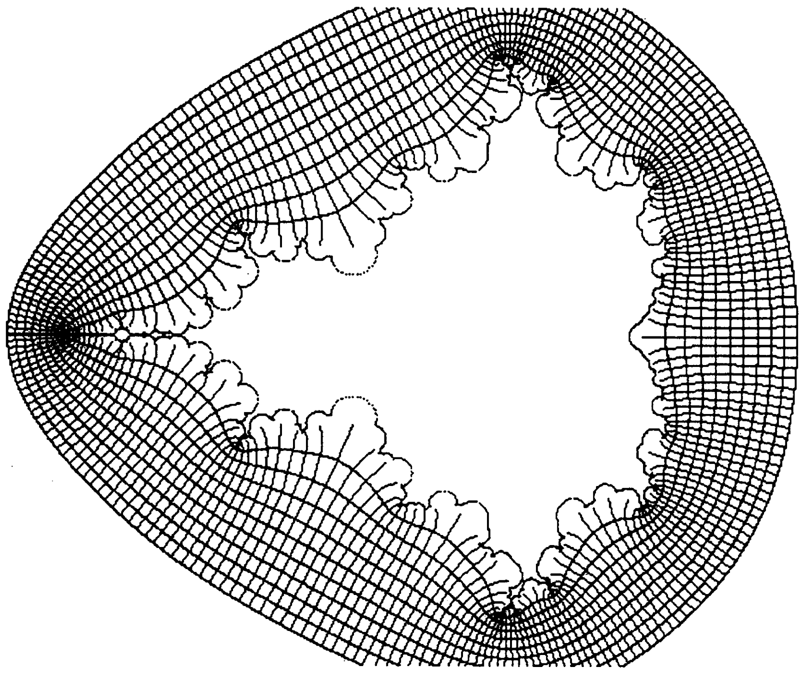

Figure 1:

3. Zero Coefficients.

Lemma 2. ([J] Theorem 5, [E-S90] Theorem 3)
For any integer $\nu \geq 2$ and any integer $K \leq 2^{\nu+1} - 5$, let $m = K2^\nu$. Then

$$b_m = 0.$$

Jungreis shows the case where $K = 1$, Ewing and Schober showed the general case.

It is conjectured by Ewing and Schober that the converse of Lemma 2 holds, namely if $b_m = 0$ then m has the form of $m = K2^\nu$ where $\nu \geq 2$ and $K \leq 2^{\nu+1} - 5$. In [B-F-H], the conjecture is confirmed for m's up to 1000. And in [E-S92], it is reported that there is no counter example up to $m = 240000$.

We shall show that the conjecture is false for higher degree dynamical systems. Now consider the system $p_{d,c}(z) = z^d + c$, where $d \geq 2$, $z, c \in \mathbb{C}$. Denote by M_d the connected locus for $p_{d,c}$. Since M_d is connected, an analytic homeomorphism $\Psi_d : \hat{\mathbb{C}} \setminus \overline{\mathbb{D}} \to \hat{\mathbb{C}} \setminus M_d$ exists. Suppose

$$\Psi_d(z) = z + \sum_{m=0}^{\infty} b_{d,m} z^{-m}.$$

We obtain the following by modifing the proof of Ewing and Schober.

Theorem 1. *Suppose $d \geq 2$. For any integers $\nu \geq 1$ and K such that $K \leq d^{\nu+1} - 3d + 1$, let $m = Kd^{\nu}$. Then*

$$b_{d,m} = 0.$$

The following theorem shows that Ewing and Schober's conjecture does not hold for $d \geq 3$.

Theorem 2. *For any degrees $d \geq 3$. For all $m \in \mathbb{N}$ such that $m \not\equiv -1 \pmod{d-1}$,*

$$b_{d,m} = 0.$$

Proof. The set M_d is symmetric under the rotation about the origin through $\frac{2\pi}{d-1}$.

Let $\omega = e^{2\pi i/(d-1)}$. Then $\omega^d = \omega$ and so, it follows that

$$\Psi_d(\omega z) = \omega \Psi_d(z).$$

Since

$$\omega \Psi_d(z) = \omega z + \omega \sum_{m=0}^{\infty} b_{d,m} z^{-m},$$

and

$$\begin{aligned} \Psi_d(\omega z) &= \omega z + \sum_{m=0}^{\infty} b_{d,m} (\omega z)^{-m} \\ &= \omega z + \sum_{m \equiv 0} b_{d,m} z^{-m} + \omega \sum_{m \equiv -1} b_{d,m} z^{-m} + \cdots + \omega^{d-2} \sum_{m \equiv 2-d} b_{d,m} z^{-m}, \end{aligned}$$

the result follows. □

The above theorem does not exhaust the whole cases where $b_{d,m} = 0$. In fact, by numerical computation, we see that $b_{3,21} = 0, b_{3,225} = 0$.

4. Nonzero Coefficients.

From the results of [E-S92], we obtain the following theorem.

Theorem 3. *For $\nu \geq 1$ and $K = 2^{\nu+1} - 3, 2^{\nu+1} - 1$, if $m = K2^{\nu}$, then*

$$b_m = -\frac{(2^{\nu+1} - 4)!}{2^K 2^{\nu}! (2^{\nu} - 2)!}.$$

For $\nu \geq 2$ and $K = 2^{\nu+1} + 1$, if $m = K2^\nu$, then

$$b_m = -\frac{3(2^\nu - 6)2^{\nu+1}!}{2^{2^{\nu+1}+4}(2^{\nu+1} - 1)(2^{\nu+1} - 5)2^\nu!(2^\nu + 1)!}.$$

The first half of this theorem shows that Zagier's conjecture holds. Namely,

Corollary 1. (D.Zagier's conjecture [B-F-H])
For $\nu \geq 1$, let the first two odd integers $m_1 < m_2$ larger than $2^{\nu+1} - 5$. Then

$$b_{m_1 2^\nu} = 4 b_{m_2 2^\nu}.$$

From the second half of the theorem, we have the following.
Corollary 2.

$$\limsup_{m \to \infty} m^{\frac{5}{4}} |b_m| \geq \frac{3 \cdot 2^{\frac{19}{4}}}{\sqrt{\pi}}.$$

Proof. An application of Wallis's formula to the last coefficients in theorem 3 leads to the asymptotics $|b_m| \sim 3/(2^{6+5\nu/2}\sqrt{\pi})$ as $\nu \to \infty$. Since $m \sim 2^{2\nu+1}$ as $\nu \to \infty$, we have $m^{5/4}|b_m| \sim 3 \cdot 2^{19/4}/\sqrt{\pi}$ as $m \to \infty$.

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